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THE MAXIMAL SIZE OF GRAPHS WITH AT MOST k EDGE-DISJOINT PATHS CONNECTING ANY TWO ADJACENT VERTICES

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Let n and k be positive integers satisfying $k+1 \leqslant n \leqslant 3k-1$, and G a simple graph of order n and size $e(G)$ with at most k edge-disjoint paths connecting any two adjacent vertices. In this paper we prove that $e(G) \leqslant \lfloor (n+k)^2/8 \rfloor$, and give complete characterizations of the extremal graphs and the extremal minimally k-edge-connected graphs.

1

The terminology used in this paper is rather standard. For the sake of clarity we include some important definitions.

All graphs under consideration are undirected and simple. A graph G consists of a nonempty set $V(G)$ of vertices and a set $E(G)$ of edges. The number of vertices of G, $|V(G)|$, is called the order of G. The number of edges of G, $|E(G)|$, is called the size of G and is denoted by $e(G)$. Let (x, y) denote the edge joining the vertices x and y. For subsets X and Y of $V(G)$, put

$$(X, Y) = \{(x, y) \in E(G) \mid x \in X, y \in Y\}.$$

If X is a singleton $\{x\}$, we write (x, Y) for $(\{x\}, Y)$. The degree of vertex x of G is denoted by $d_G(x)$.

Let $\bar{G}$ denote the complement of a graph G, and K_n the complete graph of order n. If $E' \subseteq E(G)$, we write $G - E'$ for the graph obtained from G by removing the edges in E'. Similarly, if $W \subset V(G)$, we write $G - W$ for the graph obtained from G by removing the vertices in W and all the edges incident with any vertex of W. For a singleton $X = \{x\}$, we simply write $G - x$ instead of $G - \{x\}$. For a nonempty subset X of $V(G)$, let $G[X]$ denote the subgraph induced by X.

The local edge-connectivity $\lambda(x, y)$ of two distinct vertices x and y of G is the maximum number of edge-disjoint paths connecting x and y. We define the edge-connectivity $\lambda(G)$ of a graph G to be the minimum number of edges whose removal results in a disconnected graph. A graph G is said to be minimally k-edge-connected if $\lambda(G) = k$ and $\lambda(G - e) = k - 1$ for each edge e of G. An edge cut of a graph G is a subset of $E(G)$ of the form $(X, \bar{X})$, where X is a nonempty proper subset of $V(G)$ and $\bar{X} = V(G) - X$.

Given two disjoint graphs G_1 and G_2, their join $G_1 + G_2$ is defined by $V(G_1 + G_2) = V(G_1) \cup V(G_2)$ and $E(G_1 + G_2) = E(G_1) \cup E(G_2) \cup \{(x, y) \mid x \in V(G_1),\ y \in V(G_2)\}$.

As usual, $\lfloor r \rfloor$ denotes the greatest integer $\leqslant r$. Most graphical terms and notation used in this paper may be found in [2].

2

Let us investigate the following problem of Mader [4], which is a mild variant of the first problem of this kind considered in [1].

Problem. Determine the maximal size of a graph G with $\lambda(x, y) \leqslant k$ for each edge (x, y) of G.

Madar established the following:

Theorem 1 [4]. *For a positive integer k, every graph G with order $n \geqslant k$ and $e(G) > kn - \binom{k+1}{2}$ contains an edge (x, y) with $\lambda(x, y) > k$.*

As Mader mentioned, this result is not sharp for graphs of order $n > k + 1 \geqslant 3$. However, for the graphs of high order, Mader gave the following nice answer.

Theorem 2 [4]. *For a positive integer $k \geqslant 2$, every graph $G \neq K_{k,n-k}$ with order $n \geqslant 3k$ and $e(G) \geqslant k(n - k)$ contains an edge (x, y) with $\lambda(x, y) > k$.*

In what follows we always suppose that k is a fixed integer $\geqslant 2$.

The present paper is a continuation of [3]. Its purpose is to solve Mader's problem for the remaining cases, i.e., for the graphs of order n satisfying $k + 1 \leqslant n \leqslant 3k - 1$.

In this paper we determine the maximal size $\lfloor (n + k)^2/8 \rfloor$ of a graph G of order n with $k + 1 \leqslant n \leqslant 3k - 1$ and $\lambda(x, y) \leqslant k$ for every edge (x, y) of G, and give complete characterizations of the extremal graphs for the problem and the extremal minimally k-edge-connected graphs.

3

Our first aim is to improve the lower bound given by Theorem 1.

Theorem 3. *Let G be a graph of order n satisfying $k + 1 \leqslant n \leqslant 3k - 1$. If $\lambda(x, y) \leqslant k$ for every edge (x, y) of G, then*

$$e(G) \leqslant \lfloor (n + k^2)/8 \rfloor. \tag{1}$$

Proof. We prove (1) by induction on n. For $n = k + 1$, obviously,

$$e(G) \leqslant n(n-1)/2 = \lfloor (n+k)^2/8 \rfloor,$$

i.e., (1) holds. So suppose that $n \geqslant k + 2$ and the assertion is true for smaller values of n. To complete the proof it suffices to prove the following two theorems.

Theorem 4. Let G be a graph of order n such that each edge of G is incident with at least one vertex of degree $\leqslant k$. Then (1) holds.

Proof. Let $T = \{v \in V(G) \mid d_G(v) \leqslant k\}$. Then $U = V(G) - T$ is the set of vertices of degree $\geqslant k + 1$. Put $u = |U|$. Thus $|T| = n - u$. Since each edge of G is incident with at least one vertex of T, $G[U]$ consists of u isolated vertices.

For each vertex $v \in T$, let $d'(v) = |(v, U)|$. One can easily calculate $e(G)$ as follows.

$$2e(G) = \sum_{v \in V(G)} d_G(v) = \sum_{v \in T} d_G(v) + \sum_{v \in U} d_G(v).$$

As $\sum_{v \in U} d_G(v) = \sum_{v \in T} d'(v)$,

$$2e(G) = \sum_{v \in T} [d_G(v) + d'(v)]. \tag{2}$$

Since for each vertex $v \in T$,

$$d_G(v) \leqslant k, \qquad d'(v) \leqslant u, \tag{3}$$

we obtain

$$2e(G) \leqslant (k+u)\,|T| = (k+u)(n-u). \tag{4}$$

Put

$$D = (n+k)^2 - 8e(G). \tag{5}$$

Then

$$D = (n-k-2u)^2 + 4(k+u)(n-u) - 8e(G). \tag{6}$$

In order to prove (1) it suffices to show $D \geqslant 0$, which follows from (4) and (6). □

Theorem 5. *Let G be a graph of order n with $k + 1 \leqslant n \leqslant 3k - 1$ and $\lambda(x, y) \leqslant k$ for every edge (x, y) of G. If G contains an edge (s, t) satisfying $d_G(s) \geqslant k + 1$ and $d_G(t) \geqslant k + 1$, then*

$$e(G) < \lfloor (n+k)^2/8 \rfloor. \tag{7}$$

Proof. Clearly $n \geqslant k + 3$, for otherwise $n = k + 2$ and $\lambda(s, t) = k + 1$.

By Menger's theorem [6], there exists an edge cut $C_{st} = (S, \bar{S})$ such that $s \in S$, $t \in \bar{S}$ and $|C_{st}| = \lambda(s, t) \leqslant k$. We may assume without loss of generality that $|S| \leqslant |\bar{S}|$. Put $n_1 = |S|$, $G_1 = G[S]$ and $G_2 = G(\bar{S}]$. As $d_G(s) \geqslant k + 1$ and $|C_{st}| \leqslant k$, we have $n_1 \geqslant 2$. If $n_1 = 2$, say $S = \{s, r\}$, then it is easily seen that $d_G(r) = 1$. Therefore, by the induction hypothesis of Theorem 3,

$$e(G) = e(G - r) + 1 \leqslant \lfloor (n - 1 + k)^2/8 \rfloor + 1.$$

We claim that

$$\lfloor (n - 1 + k)^2/8 \rfloor + 1 < \lfloor (n + k)^2/8 \rfloor,$$

which implies (7) holds.

In fact, if $k \geqslant 3$, then $(n + k)^2 \geqslant (n - 1 + k)^2 + 16$, as required. And if $k = 2$, then $n = 5$ since $k + 3 \leqslant n \leqslant 3k - 1$. Thus

$$\lfloor (n - 1 + k)^2/8 \rfloor = 4, \qquad \lfloor (n + k)^2/8 \rfloor = 6.$$

The claim is proved.

Therefore we may assume $n_1 \geqslant 3$. Hence $k \geqslant 3$ as $n \leqslant 3k - 1$ and $n - n_1 \geqslant n_1$. Because $|C_{st}| \leqslant k$,

$$e(G) = e(G_1) + e(G_2) + |C_{st}| \leqslant e(G_1) + e(G_2) + k. \tag{8}$$

We distinguish three cases to consider.

Case 1: $n_1 \geqslant k + 2$ *and* $n - n_1 \geqslant k + 2$.

Then, by the induction hypothesis of Theorem 3,

$$e(G_1) \leqslant \lfloor (n_1 + k)^2/8 \rfloor, \qquad e(G_2) \leqslant \lfloor (n - n_1 + k)^2/8 \rfloor.$$

It follows from (8) that

$$e(G) \leqslant (n_1 + k)^2/8 + (n - n_1 + k)^2/8 + k = (n + k)^2/8 + f_1/8,$$

where $f_1 = k^2 + 8k + 2n_1^2 - 2n_1 n$. Since $n - n_1 \geqslant k + 2$ and $n_1 \geqslant k + 2$,

$$\begin{aligned} f_1 &\leqslant k^2 + 8k + 2n_1^2 - 2n_1(n_1 + k + 2) = k^2 + 8k - 2n_1(k + 2) \\ &\leqslant k^2 + 8k - 2(k + 2)^2 < -8. \end{aligned}$$

Consequently (7) holds for Case 1.

Case 2: $n_1 \leqslant k + 1$ *and* $n - n_1 \geqslant k + 2$.

Then

$$e(G_1) \leqslant n_1(n_1 - 1)/2, \qquad e(G_2) \leqslant \lfloor (n - n_1 + k)^2/8 \rfloor.$$

By (8)

$$\begin{aligned} e(G) &\leqslant n_1(n_1 - 1)/2 + (n - n_1 + k)^2/8 + k \\ &= (n + k)^2/8 + [5n_1^2 - 2n_1(n + k + 2) + 8k]/8 \\ &= (n + k)^2/8 + f_2/8, \end{aligned}$$

where $f_2 = 5n_1^2 - 2n_1(n+k+2) + 8k$. Since $n - n_1 \geq k+2$,

$$\begin{aligned} f_2 &= 5n_1^2 - 2n_1(n+k+2) + 8k \\ &\leq 5n_1^2 - 2n_1(n_1+2k+4) + 8k = 3n_1^2 - 4n_1(k+2) + 8k. \end{aligned}$$

It is easily seen that for $3 \leq n_1 \leq k+1$

$$\begin{aligned} g(n_1) &= 3n_1^2 - 4n_1(k+2) + 8k \leq \max\{g(3), g(k+1)\} \\ &= \max\{3-4k, 2k-k^2-5\} \leq -8, \end{aligned}$$

since $k \geq 3$. So $f_2 \leq -8$, as required.

Case 3: $n_1 \leq k+1$ and $n - n_1 \leq k+1$.

Then (8) yields

$$\begin{aligned} e(G) &\leq n_1(n_1-1)/2 + (n-n_1)(n-n_1-1)/2 + k \\ &= [n^2 - n(2n_1+1) + 2n_1^2 + 2k]/2 = (n+k)^2/8 + f_3/8, \end{aligned} \tag{9}$$

where

$$f_3 = 3n^2 - 2n(4n_1+k+2) + 8n_1^2 - k^2 + 8k. \tag{10}$$

We separate this case into three subcases.

Subcase 1: $n_1 \leq k$ and $n - n_1 \leq k$. Then, considering $n \geq k+3$ and $k \geq 3$, we have

$$\begin{aligned} f_3 &= 8(k-n_1)(n-n_1-k) - (2k-n)[3(n-k)-4] - k(n-k) \\ &\leq -k(n-k) \leq -9, \end{aligned}$$

from which (7) follows.

Subcase 2: $n_1 \leq k$ and $n - n_1 = k+1$. Then, on the substitution of $n = n_1 + k + 1$ into (10), we obtain

$$f_3 = 3n_1^2 - 4kn_1 - 6n_1 + 8k - 1. \tag{11}$$

There exist two possibilities.

(a). $k \geq n_1 + 1$. Then $f_3 = -8 - (n_1-3)(n_1+5) - 4(k-n_1-1)(n_1-2) \leq -8$ since $n_1 \geq 3$. Hence (7) holds.

(b). $k = n_1$. Then $f_3 = -(n_1-1)^2$. We claim that (7) holds. For otherwise

$$e(G) = \lfloor (n+k)^2/8 \rfloor$$

as $f_3 \leq 0$. By (9)

$$e(G) = k(k-1)/2 + k(k+1)/2 + k = \lfloor (n+k)^2/8 \rfloor,$$

implying $G_2 = K_{k+1}$. Because $d_G(s) \geq k+1$ and $n_1 = k$, there exist two distinct vertices w and z in $\bar{S}$ adjacent to s. Consequently $\lambda(w, z) \geq k+1$ for edge (w, z) since G_2 contains k edge-disjoint paths connecting w and z. We arrive at a desired contradiction.

Subcase 3: $n_1 = k+1$ and $n - n_1 = k+1$. By substituting $n_1 = k+1$ into (11), one gets $f_3 = -(k-2)^2$. Supposing (7) did not hold, we should have

$$e(G) = \lfloor (n+k)^2/8 \rfloor,$$

since $f_3 < 0$. A similar argument used above shows that

$$e(G) = k(k+1)/2 + k(k+1)/2 + k = \lfloor (n+k)^2/8 \rfloor.$$

Therefore

$$|C_{st}| = k, \qquad G_1 = G_2 = K_{k+1}.$$

We leave it to the reader to show that either in $E(G_1)$ or in $E(G_2)$ there exists an edge (w, z) with $\lambda(w, z) \geqslant k+1$, which is a desired contradiction. The proof is complete. □

4

Our next aim is to characterize the extremal graphs, i.e., the graphs of order n and size $\lfloor (n+k)^2/8 \rfloor$ with $k+1 \leqslant n \leqslant 3k-1$ and $\lambda(x, y) \leqslant k$ for any two adjacent vertices x and y.

Let $\mathbb{B}_1(p, r)$ denote the set of all r-regular graphs with order p, and $\mathbb{B}_2(p, r)$ the set of all graphs of order p such that a specified vertex v_0 has degree $r+1$ and every other vertex has degree r.

For convenience sake, it is stipulated that $\bar{K}_u + H = H$ for any graph H if $u = 0$. Now let us define two sets $\mathbb{A}_1(n, k)$ and $\mathbb{A}_2(n, k)$ of graphs with order n such that $k+1 \leqslant n \leqslant 3k-1$ as follows.

$$\mathbb{A}_1(n, k) = \{\bar{K}_u + H \mid H \in \mathbb{B}_1(n-u, k-u)\}$$

where $u = (n-k)/2$ if $n+k \equiv 0 \pmod 4$, $u = ((n-k)/2) \pm 1$ if $n+k \equiv 2 \pmod 4$ (when $n = k+2$, take '$-$' only), and $u = (n-k \pm 1)/2$ if $n+k \equiv 1 \pmod 2$ (when $n = k+1$, take '$-$' only).

Provided $n+k \equiv 2 \pmod 4$,

$$\mathbb{A}_2(n, k) = \{\bar{K}_u + H - e \mid H \in \mathbb{B}_2(n-u, k-u),\ e \in E_0\}$$

where $u = (n-k)/2$; E_0 is the set of edges of $\bar{K}_u + H$ incident with the specified vertex v_0 of H with $d_H(v_0) = k-u+1$, and E_0 is restricted further within $E(H)$ when $n = k+2$.

Proposition 1. $A_i(n, k) \neq \emptyset$, $i = 1, 2$.

Proof. To prove $A_1(n, k) \neq \emptyset$ it suffices to show $B_i(n-u, k-u) \neq \emptyset$, i.e., $(n-u)(k-u) \equiv 0 \pmod 2$ if $i = 1$ and $(n-u)(k-u) \equiv 1 \pmod 2$ if $i = 2$.

First let us prove $\mathbb{A}_1(n, k) \neq \emptyset$.

If $n+k \equiv 0 \pmod 4$, then, by the definition of $\mathbb{A}_1$(n, k),

$$u = (n-k)/2.$$

Thus

$$n - u = (n+k)/2 \equiv 0 \pmod 2.$$

If $n+k\equiv 2 \pmod 4$, then

$$u=((n-k)/2)\pm 1, \qquad n-u=((n+k)/2)\mp 1\equiv 0 \pmod 2.$$

If $n+k\equiv 1 \pmod 2$, then $n-u+k-u\equiv 1 \pmod 2$, which implies

$$(n-u)(k-u)\equiv 0 \pmod 2.$$

Suppose then that $n+k\equiv 2 \pmod 4$, we prove $\mathbb{A}_2(n,k)\neq\emptyset$.

Since $u=(n-k)/2$ and $n+k\equiv 2 \pmod 4$, we have

$$n-k=(n+k)/2\equiv 1 \pmod 2, \qquad k-u=(3k-n)/2\equiv 1 \pmod 2.$$

Thus $(n-u)(k-u)\equiv 1 \pmod 2$, yielding the desired assertion. □

Proposition 2. *If $G\in\mathbb{A}_1(n,k)\cup\mathbb{A}_2(n,k)$, then $e(G)=\lfloor (n+k)^2/8\rfloor$ and $\lambda(x,y)\leq k$ for every edge (x,y) of G.*

Proof. As every edge of G is incident with at least one vertex of degree $\leq k$, $\lambda(x,y)\leq k$ for every edge (x,y) of G. To prove

$$e(G)=\lfloor (n+k)^2/8\rfloor$$

it suffices to show

$$D=(n+k)^2-8e(G)\leq 7. \tag{12}$$

If $G\in\mathbb{A}_1(n,k)$, then $d_G(v)=n-u$ for each vertex $v\in V(\bar{K}_u)$ and $d_G(v)=k$ for each vertex $v\in V(H)$. Therefore

$$2e(G)=k(n-u)+u(n-u)=(k+u)(n-u),$$

yielding $D=(n+k)^2-8e(G)=(n-k-2u)^2$. It is easy to check that for all the values of u defining $\mathbb{A}_1(n,k)$

$$|n-k-2u|\leq 2,$$

from which (12) follows,

Similarly, if $G\in\mathbb{A}_2(n,k)$, then $2e(G)=(k+u)(n-u)-1$. By using $u=(n-k)/2$, we obtain

$$D=(n+k)^2-8e(G)=(n-k-2u)^2+4=4,$$

as required. □

Remark. The upper bound $\lfloor (n+k)^2/8\rfloor$ for $e(G)$ given by Theorem 3 is best possible, i.e., given $k\geq 2$, for each n satisfying $k+1\leq n\leq 3k-1$ there exists at least one graph in question with order n and size $\lfloor (n+k)^2/8\rfloor$ even though the graph is restricted within $\mathbb{A}_1(n,k)$.

Theorem 6. *Let G be a graph of order n with $k+1\leq n\leq 3k-1$ and $\lambda(x,y)\leq k$ for every edge (x,y) of G. Then, up to isomorphism, $G\in\mathbb{A}_1(n,k)\cup\mathbb{A}_2(n,k)$ provided $e(G)=\lfloor (n+k)^2/8\rfloor$.*

Proof. By Theorem 5, every edge of G is incident with at least one vertex of degree $\leqslant k$, i.e., G satisfies the conditions of Theorem 4. Let us keep the notation of the proof of Theorem 4.

By combining (2), (3) and (6), we have

$$e(G) \leqslant (k+u)(n-u)/2 \leqslant (n+k)^2/8.$$

As $e(G) = \lfloor (n+k)^2/8 \rfloor$, one obtains

$$D = (n+k)^2 - 8e(G) \leqslant 7, \qquad 0 \leqslant (k+u)(n-u) - 2e(G) \leqslant 1.$$

Put $d = (k+u)(n-u) - 2e(G)$. Thus $0 \leqslant d \leqslant 1$. Therefore

$$\begin{aligned} D &= (n+k)^2 - 4(k+u)(n-u) + 4d \\ &= (n-k-2u)^2 + 4d \leqslant 7. \end{aligned} \tag{13}$$

We distinguish two cases according to $d = 0$ or 1.

Case 1: $d = 0$.

Then $(k+u)(n-u) = 2e(G)$, implying by (3) that for each vertex $v \in T$

$$d_G(v) = k, \qquad d'(v) = u.$$

On the other hand by (13), $(n-k-2u)^2 \leqslant 4$, i.e.,

$$(n-k)/2 - 1 \leqslant u \leqslant (n-k)/2 + 1. \tag{14}$$

We examine three subcases.

Subcase 1: $n + k \equiv 0 \pmod 4$. Then

$$n \equiv k \pmod 2, \tag{15}$$

$$(n-k)/2 \equiv k \pmod 2. \tag{16}$$

Seeing $(k+u)(n-u) = 2e(G) \equiv 0 \pmod 2$, we deduce from (15) and (16) that

$$u \equiv k \equiv (n-k)/2 \pmod 2.$$

By (14),

$$u = (n-k)/2.$$

It follows that $G \in \mathbb{A}_1(n, k)$.

Subcase 2: $n + k \equiv 2 \pmod 4$. Then

$$n \equiv k \pmod 2, \tag{17}$$

$$(n-k)/2 \equiv k + 1 \pmod 2. \tag{18}$$

As $(k+u)(n-u) = 2e(G) \equiv 0 \pmod 2$, (17) yields $u \equiv k \pmod 2$. So, by (14) and (18),

$$u = (n-k)/2 \pm 1. \tag{19}$$

When $n = k + 2$, we should take only '$-$' in (19). For otherwise $d_G(v) \leqslant n - u \leqslant k$ for $v \in U$. Thus $G \in \mathbb{A}_1(n, k)$.

Subcase 3: $n + k \equiv 1 \pmod 2$. Then $n - k \equiv 1 \pmod 2$, implying by (14)

$$u = (n - k \pm 1)/2.$$

Obviously, $u = 0$ when $n = k + 1$. Consequently $G \in \mathbb{A}_1(n, k)$.

Case 2: $d = 1$.

Then

$$(k + u)(n - u) = 2e(G) + 1. \tag{20}$$

From (13), $(n - k - 2u)^2 \leqslant 1$, yielding

$$(n - k - 1)/2 \leqslant u \leqslant (n - k + 1)/2.$$

By (20), $n \equiv k \equiv u + 1 \pmod 2$. Hence

$$u = (n - k)/2, \qquad n + k \equiv 2 \pmod 4.$$

It follows from (20) that $d_G(v) = k$ and $d'(v) = u$ for every vertex $v \in T$ other than v_0, and either $d_G(v_0) = k - 1$, $d'(v_0) = u$ or $d_G(v_0) = k$, $d'(v_0) = u - 1$. So $G \in \mathbb{A}_2(n, k)$.

The proof is complete. □

We summarize the results obtained above in the following

Theorem 7. *Let G be a graph of order n such that $k + 1 \leqslant n \leqslant 3k - 1$. If $G \notin \mathbb{A}_1(n, k) \cup \mathbb{A}_2(n, k)$, then either $e(G) < \lfloor (n + k)^2/8 \rfloor$ or G contains an edge (x, y) with $\lambda(x, y) > k$.*

5

Finally we characterize the extremal minimally k-edge-connected graphs.

Let $\mathbb{A}_1(n, k)$ and $\mathbb{B}_2(n, k)$ be defined as above. We modify $\mathbb{A}_2(n, k)$ as follows. On condition that $n \geqslant k + 4$ and $n + k \equiv 2 \pmod 4$, set

$$\mathbb{A}_2'(n, k) = \{\bar{K}_u + H - e \mid H \in \mathbb{B}_2(n - u, k - u),\ e \in E_0'\}$$

where $u = (n - k)/2$; E_0' is the set of edges of $\bar{K}_u + H$ joining the specified vertex v_0 of H and the vertices of $\bar{K}_u$.

Proposition 3. *If $G \in \mathbb{A}_1(n, k) \cup \mathbb{A}_2'(n, k)$, then G is minimally k-edge-connected and*

$$e(G) = \lfloor (n + k)^2/8 \rfloor.$$

Proof. Since $\mathbb{A}_2'(n, k) \subseteq \mathbb{A}_2(n, k)$, we have $G \in \mathbb{A}_1(n, k) \cup \mathbb{A}_2'(n, k) \subseteq \mathbb{A}_1(n, k) \cup \mathbb{A}_2(n, k)$. By Proposition 2

$$e(G) = \lfloor (n + k)^2/8 \rfloor.$$

A similar argument used in the proof of Theorem 5 shows that G contains no edge cut $(S, \bar{S})$ such that $|(S, \bar{S})| < k$ and $2 \leq |S| \leq n-2$. Therefore $\lambda(G) = \min\{d_G(v) \mid v \in V(G)\} = k$. Clearly, G is minimally k-edge-connected since each edge of G is incident with at least one vertex of degree k. □

An easy consequence of Theorems 3 and 6 is:

Theorem 8 [7]. *Let G be a minimally k-edge-connected graph with order n satisfying $k+1 \leq n \leq 3k-1$. Then*

$$e(G) \leq \lfloor (n+k)^2/8 \rfloor .$$

Moreover, the equality holds only if $G \in \mathbb{A}_1(n, k) \cup \mathbb{A}_2'(n, k)$.

The easy proof is left to the reader.

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